(1) $$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2}\frac{X_i^2}{\theta}\right) = \left(\frac{1}{2\pi\theta}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\theta}\sum_{i=1}^{n} X_i^2\right)$$

$$\ell(\theta) = -\frac{n}{2}\log 2\pi\theta - \frac{1}{2\theta}\sum_{i=1}^{n} X_i^2$$

$$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^{n} X_i^2$$

$$\ell'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$$

The MLE $\hat{\theta}$ solves $\ell'(\hat{\theta}) = 0$.

$$-\frac{n}{2\hat{\theta}} + \frac{1}{2\hat{\theta}^2}\sum_{i=1}^{n} X_i^2 = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 .$$

It is also a method of moments estimator of θ because $X_i \sim N(0,\theta)$ implies $\mathbb{E}(X_i^2) = Var(X_i) + (\mathbb{E}(X_i))^2 = \theta$. Replacing population averages with sample averages gives $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \hat{\theta}_{MME}$.

(2) Approach 1:

The law of large numbers allows us to conclude that whenever $X_1, \ldots, X_n$ is IID, and $\mathbb{E}(X_i) < \infty$, then $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} \mathbb{E}(X_i)$.

Since $X_1, \ldots, X_n$ are IID, $X_1^2, \ldots, X_n^2$ are also IID. $\mathbb{E}(X_i^2) = \theta < \infty$. By the law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} \mathbb{E}(X_i^2) = \theta$. Therefore, $\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is consistent for θ.

Approach 2: $X_i \sim N(0,\theta)$ for all i

$$\mathbb{E}(\hat{\theta}_{MLE}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2) \overset{\downarrow}{=} \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta.$$

Thus, $\hat{\theta}_{MLE}$ is unbiased for θ.

$$Var(\hat{\theta}_{MLE}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) \underset{\uparrow}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = \frac{1}{n^2}\sum_{i=1}^{n}\left[\mathbb{E}(X_i^4) - (\mathbb{E}(X_i^2))^2\right]$$

independence of $X_1, \dots, X_n$

given $\mathbb{E}(X_i^4) = 3\theta^2$

$$\overset{\downarrow}{=} \frac{1}{n^2}\sum_{i=1}^{n}(3\theta^2 - \theta^2) = \frac{2\theta^2}{n}$$

Since $\lim_{n\to\infty}\left[\hat{\theta}_{MLE} - \theta\right] = 0$, $\hat{\theta}_{MLE}$ is asymptotically unbiased for θ. Since $\lim_{n\to\infty}\frac{2\theta^2}{n} = 0$, then $Var(\hat{\theta}_{MLE}) \to 0$ as $n\to\infty$. Therefore, $\hat{\theta}_{MLE}$ is squared-error consistent for θ. ~~Squared~~

(You can also use Chebyshev's inequality here),

$$\mathbb{P}(|\hat{\theta}_{MLE} - \underset{\mathbb{E}(\hat{\theta}_{MLE})}{\underset{\shortparallel}{\theta}}| \geq \varepsilon) \leq \frac{Var(\hat{\theta}_{MLE})}{\varepsilon^2} = \frac{2\theta^2}{n\varepsilon^2}$$

$$\lim_{n\to\infty}\mathbb{P}(|\hat{\theta}_{MLE} - \theta| \geq \varepsilon) = 0.$$

Thus, $\hat{\theta}_{MLE}$ is consistent for θ.

(3)

$$\ell''(\theta) = +\frac{n}{2\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^{n} X_i^2$$

$$I(\theta) = \mathbb{E}(-\ell''(\theta)) = -\frac{n}{2\theta^2} + \frac{1}{\theta^3}\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right) = \frac{-n}{2\theta^2} + \frac{1}{\theta^3}n\theta$$

$$= -\frac{n}{2\theta^2} + \frac{n}{\theta^2} = \frac{n}{2\theta^2}$$

$$\left[I(\theta)\right]^{-1} = \frac{2\theta^2}{n} = Var(\hat{\theta}_{MLE}).$$

Thus, $\hat{\theta}_{MLE}$ is an efficient estimator of θ.

<u>NOTE</u> You can also do ~~the~~ a per-unit calculation if you want. But it saves time to use what is already available in (1).

(4) $\tilde{\theta} = X_1^2$ is not sufficient for θ.

~~We can look at the conditional distribution of~~

Take two datasets which have the same value for $\tilde{\theta}$. For example, $X_1^{(1)} = 1$, $X_2^{(1)} = 2$ and $X_1^{(2)} = 1$, $X_2^{(2)} = 3$. Both datasets of size 2 give the same observed value for $\tilde{\theta}$.

$$\frac{L(\theta; X_1^{(1)}, X_2^{(1)})}{L(\theta; X_1^{(2)}, X_2^{(2)})} = \frac{\frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2}\frac{1}{\theta}\right) \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2}\frac{2}{\theta}\right)}{\frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2}\frac{1}{\theta}\right) \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2}\frac{3}{\theta}\right)}$$

$$= \exp\left(-\frac{1}{2\theta}(2-3)\right) = \exp\left(-\frac{1}{2\theta}\right)$$

The ratio of the likelihoods for these two datasets of the same size and the same observed value for $\tilde{\theta}$ still depends on θ. Thus, $\tilde{\theta} = X_1^2$ is not sufficient for θ.

<u>NOTE</u> You will find it extremely difficult to use the factorization theorem here to show that $\tilde{\theta}$ is not sufficient for θ. You need to go through all possible ways of splitting up the likelihood $L(\theta)$ into different parts and show that <u>NONE</u> of them work.

<u>Exercise</u> Perhaps you could also think of a sufficient statistic for θ. If you think of one, think of another and explain why.

(5) under the null $\theta = 1$, the maximized likelihood is

$$L(1) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} X_i^2\right)$$

Under the alternative $\theta \neq 1$, the maximized likelihood can be found after finding the solution to

$$\max_{\theta \neq 1} L(\theta).$$

The maximizer here is $\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Thus,

$$L(\hat{\theta}_{MLE}) = \left(\frac{1}{2\pi\hat{\theta}_{MLE}}\right)^{n/2} \exp\left(-\frac{1}{2\hat{\theta}_{MLE}}\sum_{i=1}^{n} X_i^2\right)$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{-n/2} \exp\left(-\frac{n}{2}\right)$$

The generalized likelihood ratio is

$$\frac{L(1)}{L(\hat{\theta}_{MLE})} = \frac{\left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} X_i^2\right)}{\left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{-n/2} \exp\left(-\frac{n}{2}\right)}$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{n/2} \exp\left(-\frac{n}{2}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1\right)\right)$$

NOTE To find the integral which needs to be evaluated, we need the distribution of $\frac{1}{n}\sum_{i=1}^{n} X_i^2$. This takes a lot more work. But the idea is to find a statistic on which the likelihood depends on for which the distribution is known. Return to the uniform example for more.

But what you need is a critical region at level $\alpha = 0.01$

$$\mathbb{P}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{n/2} \exp\left(-\frac{n}{2}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1\right)\right) \leq \theta^* \,\middle|\, \theta = 1\right) = 0.01.$$

where θ^* is the critical value.