In-class exercise, 2023.05.11

- Write down you and your partner's ID number.
- The grading is as follows: If you did not even try or are doing something else instead of this exercise, then you get 0 automatically and will be marked absent.

Consider independent samples from three normal populations:

$$X_1, \ldots, X_n \sim N\left(\mu_1, \sigma^2\right),\ Y_1, \ldots, Y_m \sim N\left(\mu_2, \sigma^2\right),\ Z_1, \ldots, Z_l \sim N\left(\mu_3, \sigma^2\right)$$

All four parameters $\left(\mu_1, \mu_2, \mu_3, \sigma^2\right)$ are unknown. Your task is to develop a test of the null hypothesis that $H_0 : \ \mu_1 = \mu_2 = \mu_3$. Let $\left(\widehat{\mu}, \widehat{\sigma}_0^2\right)$ be the MLEs for the common population mean and variance under H_0 . Let $\left(\widehat{\mu}_1, \widehat{\mu}_2, \widehat{\mu}_3, \widehat{\sigma}^2\right)$ be the MLEs for the each population mean and the common population variance.

1. Describe the sets Θ_0 and Θ_1. Construct the generalized likelihood ratio for testing H_0.
2. Show that the generalized likelihood ratio test for testing H_0 at the $100\alpha\%$ level leads to a decision rule of the form: Reject H_0 when
$$\frac{\widehat{\sigma}_0^2}{\widehat{\sigma}^2} \geq c,$$
where c is some non-random threshold.
3. Show that
$$\sum_{i=1}^{n} (X_i - \widehat{\mu})^2 = \sum_{i=1}^{n} (X_i - \widehat{\mu}_1)^2 + n\left(\widehat{\mu}_1 - \widehat{\mu}\right)^2.$$
A similar result applies to the Y's and Z's.
4. Show that the decision rule can be rewritten in terms of
$$\frac{\widehat{\sigma}_0^2 - \widehat{\sigma}^2}{\widehat{\sigma}^2} = \frac{n\left(\widehat{\mu}_1 - \widehat{\mu}\right)^2 + m\left(\widehat{\mu}_2 - \widehat{\mu}\right)^2 + l\left(\widehat{\mu}_3 - \widehat{\mu}\right)^2}{\displaystyle\sum_{i=1}^{n} (X_i - \widehat{\mu}_1)^2 + \sum_{i=1}^{m} (Y_i - \widehat{\mu}_2)^2 + \sum_{i=1}^{l} (Z_i - \widehat{\mu}_3)^2}$$
5. What is the distribution of
$$\frac{1}{\sigma^2}\left[\sum_{i=1}^{n} (X_i - \widehat{\mu}_1)^2 + \sum_{i=1}^{m} (Y_i - \widehat{\mu}_2)^2 + \sum_{i=1}^{l} (Z_i - \widehat{\mu}_3)^2\right]?$$
Explain your finding. Is your finding dependent on whether the null is true or not?
6. Show that
$$\begin{aligned} & \frac{1}{\sigma^2}\left[n\left(\widehat{\mu}_1 - \widehat{\mu}\right)^2 + m\left(\widehat{\mu}_2 - \widehat{\mu}\right)^2 + l\left(\widehat{\mu}_3 - \widehat{\mu}\right)^2\right] \\ = \ & \frac{1}{\sigma^2}\left[n\left(\widehat{\mu}_1 - \mu\right)^2 + m\left(\widehat{\mu}_2 - \mu\right)^2 + l\left(\widehat{\mu}_3 - \mu\right)^2 - (n+m+l)\left(\widehat{\mu} - \mu\right)^2\right] \end{aligned}$$
7. Let
$$W_1 = \frac{\widehat{\mu}_1 - \mu}{\sigma/\sqrt{n}},\ W_2 = \frac{\widehat{\mu}_2 - \mu}{\sigma/\sqrt{m}},\ W_3 = \frac{\widehat{\mu}_3 - \mu}{\sigma/\sqrt{l}}.$$
What is the joint distribution of (W_1, W_2, W_3)? Consider an orthogonal transformation of the form
$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \sqrt{\dfrac{n}{n+m+l}} & \sqrt{\dfrac{m}{n+m+l}} & \sqrt{\dfrac{l}{n+m+l}} \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix}.$$
What is V_1^2? What would be the distribution of
$$\frac{1}{\sigma^2}\left[n\left(\widehat{\mu}_1 - \widehat{\mu}\right)^2 + m\left(\widehat{\mu}_2 - \widehat{\mu}\right)^2 + l\left(\widehat{\mu}_3 - \widehat{\mu}\right)^2\right]?$$
8. Complete the formulation of the decision rule by constructing an appropriate test statistic which will have a known distribution.